

A note on Griffiths infinitesimal invariant for curves

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Abstract

Given a generic curve of genus $g \geq 4$ and a smooth point $L \in W_{g-1}^1(C)$, whose linear system is base-point free, we consider the Abel-Jacobi normal function associated to $L^{\otimes 2} \otimes \omega_C^{-1}$, when (C, L) varies in moduli. We prove that its infinitesimal invariant reconstruct the couple (C, L) . When $g = 4$, we obtain the generic Torelli theorem proved by Griffiths.

1 Introduction

The infinitesimal invariant of normal function was introduced by Griffiths in [Grif83]. In that paper he gave a beautiful application of it, which we briefly recall.

If C is a generic curve of genus 4, then its canonical model is the intersection of the smooth quadric Q with a cubic in \mathbb{P}^3 . The rulings of Q cut on the canonical curve two complete g_3^1 's, $|L|$ and $|L'|$, which are adjoint, i.e. $L' = \omega_C \otimes L^{-1}$. Since the difference $L \otimes (L')^{-1}$ has degree zero, when C varies in moduli we get a normal function ν given by the Abel-Jacobi map. Griffiths proved that the infinitesimal invariant of ν gives the equation of the canonical curve inside Q .

The infinitesimal invariant was then studied by M. Green [G89] and C. Voisin [Voi88]. Green refined Griffiths' original idea defining a series of infinitesimal invariants and obtained a result on the Abel-Jacobi map of odd-dimensional projective hypersurfaces of large degree. Voisin instead gave a geometric interpretation of the infinitesimal invariant in terms of algebraic cycles.

A very interesting application of the infinitesimal invariant to the study of algebraic cycles is due to A. Collino and G.P. Pirola [CP95]. They computed the infinitesimal invariant of the Ceresa cycle $C - C^-$ of a curve in its Jacobian. As consequences, they reproved that $C - C^-$ is not algebraically trivial (originally proved by G. Ceresa [C83]) and obtained a

generic Torelli theorem for curves of genus 3. In particular they showed that the infinitesimal invariant gives the equation of the canonical curve in the projective plane.

The works mentioned above show that the infinitesimal invariant is a powerful tool in the study of algebraic cycles and in Torelli-type problems. It encodes in fact some transcendental and algebraic information at the same time.

In this paper we give the following generalisation of Griffiths result for curves.

Let C be a generic curve of genus $g \geq 4$. Let's consider a non singular point $L \in W_{g-1}^1(C)$ whose linear system is base-point free. Then also the adjoint bundle $\omega_C \otimes L^{-1} \in W_{g-1}^1(C)$ is a smooth point. The image of the Petri map $H^0(L) \otimes H^0(K_C \otimes L^{-1}) \rightarrow H^0(\omega_C)$ is a four dimensional vector subspace V of $H^0(C, \omega_C)$ whose linear system is base-point-free. The image of the associated holomorphic map $C \rightarrow \mathbb{P}V^* \cong \mathbb{P}^3$ is contained in a quadric Q of rank four and is birational to C . When we deform the couple (C, L) , the 0-degree line bundle $L^{\otimes 2} \otimes \omega_C^{-1}$ gives a normal function ν (see Section 3).

We show that the infinitesimal invariant of ν reconstruct the curve $\phi(C)$ (Proposition 3.7) and, as consequence, the couple (C, L) (Corollary 3.10).

The paper consists of two sections. The first contains a quick review of the definition of the infinitesimal invariant and of the results by Griffiths and Voisin that we need in our computations. In the second we compute the infinitesimal invariant of the normal function.

2 Review of the infinitesimal invariant

In this section we recall the basic facts on the infinitesimal invariant of normal functions for curves. A great reference is [Voi03, chap. 7].

Consider a smooth curve C and its Kuranishi family $\pi : \mathcal{C} \rightarrow B$. We define $C_t = \pi^{-1}(t)$ and $C = C_0$ for a reference point $0 \in B$. We have the associated jacobian fibration

$$j(\pi) : \frac{\mathcal{H}}{\mathcal{F} + R^1\pi_*\mathbb{Z}} \rightarrow B,$$

where \mathcal{H} and \mathcal{F} are the holomorphic vector bundles over B with fibers $\mathcal{H}_t = H^1(C_t, \mathbb{C})$ and $\mathcal{F}_t = H^0(C_t, \omega_{C_t})$ over $t \in B$.

Suppose to have a curve $\mathcal{D} \subset \mathcal{C}$ such that for every $t \in B$ the intersection divisor $D_t = \mathcal{D} \cdot C_t$ has degree zero on C_t . Then we can define a normal function

$$\nu : B \rightarrow \frac{\mathcal{H}}{\mathcal{F} + R^1\pi_*\mathbb{Z}}$$

setting $\nu(t) = AJ_{C_t}(D_t)$, where AJ is the Abel-Jacobi map.

Following Mark Green [G89] we define the infinitesimal invariant of ν in the following way.

The Gauss Manin connection ∇ of \mathcal{H} induces a morphism of vector bundles

$$\nabla : \mathcal{F} \rightarrow \mathcal{H}^{0,1} \otimes \Omega_B^1,$$

where $\mathcal{H}^{0,1} = \frac{\mathcal{H}}{\mathcal{F}}$ (note that Serre duality induces an isomorphism $\mathcal{H}^{0,1} \cong \mathcal{H}^*$).

If $\tilde{\nu} : U \subset B \rightarrow \mathcal{H}$ is a local lifting of ν , the class

$$[\nabla \tilde{\nu}] \in \frac{\mathcal{H}^{1,0} \otimes \Omega_U^1}{\nabla \mathcal{F}}$$

does not depend on the chosen lifting (see [G89]). We denote by $\delta\nu$ this class and call it the infinitesimal invariant. It is useful to define also the dual version of $\delta\nu$. In fact, let's consider the transpose of the Gauss-Manin, $\nabla^t : \mathcal{F} \otimes T_U \rightarrow \mathcal{H}^{0,1}$. Choose a reference point $0 \in U$ and denote $C = C_0$. Then $\nabla^t : H^0(C, \omega_C) \otimes H^1(C, T_C) \rightarrow H^1(C, \mathcal{O}_C)$ is given by $\nabla^t(\sum_i \omega_i \otimes \xi_i) = \sum_i \nabla_{\xi_i} \omega_i$. By the duality $\frac{\mathcal{H}^{0,1} \otimes \Omega_U^1}{\nabla \mathcal{F}} \cong (\ker(\nabla^t))^*$, we can consider $\delta\nu$ as an element of this last vector space. Over the point 0 we have:

$$\delta\nu(0)(\sum_i \omega_i \otimes \xi_i) = \sum_i \int_C \nabla_{\xi_i} \tilde{\nu} \wedge \omega_i,$$

where $\sum_i \xi_i \cdot \omega_i = 0$. This is Griffiths' definition of the infinitesimal invariant [Grif83].

The first tool in our computation will be the following

Theorem 2.1. [Grif83, pp. 292-293] *Let $\xi \otimes \omega \in H^1(C, T_C) \otimes H^0(C, \omega_C)$ be such that $\xi \cdot \omega = 0$ in $H^1(C, \mathcal{O}_C)$. Choose $h \in C^\infty(C)$ such that $\xi \otimes \omega = \bar{\partial} h$ and call $D_0 = \sum_{i=1}^l p_i - q_i$. Then the number $\sum_{i=1}^l h(p_i) - h(q_i)$ depends only on $\xi \otimes \omega$ and we have*

$$\delta\nu(0)(\xi \otimes \omega) = \sum_{i=1}^l h(p_i) - h(q_i).$$

The second tool we need is a computation by Voisin [Voi88] which we now describe in our particular case.

Suppose to have a smooth family of curves $\pi : \mathcal{C} \rightarrow \Delta$ over the unit disc in \mathbb{C} and let \mathcal{D} be as above. There is a short exact sequence over C :

$$0 \rightarrow \mathcal{O}_C \rightarrow \Omega_{\mathcal{C}|C}^1 \otimes \pi^* T_{\Delta,0} \rightarrow \omega_C \otimes \pi^* T_{\Delta,0} \rightarrow 0,$$

which gives in cohomology

$$H^0(C, \Omega_{\mathcal{C}|C}^1 \otimes \pi^* T_{\Delta,0}) \xrightarrow{\gamma} H^0(C, \omega_C) \otimes T_{\Delta,0} \xrightarrow{\delta} H^1(C, \mathcal{O}_C).$$

If $\omega \otimes \eta \in H^0(C, \omega_C) \otimes T_{\Delta,0}$ is such that $\delta(\omega \otimes \eta) = 0$, then there exists $\Omega = \hat{\omega} \otimes \eta \in H^0(C, \Omega_{C|C}^1 \otimes \pi^* T_{\Delta,0})$ such that $\gamma(\Omega) = \omega \otimes \eta$.

Write $D_0 = D_{1,0} - D_{2,0} = \sum_{i=1}^l p_i - \sum_{i=1}^l q_i$, and consider, for $i = 1, 2$, the image of Ω under the map $F_i : H^0(C, \Omega_{C|C}^1 \otimes \pi^* T_{\Delta,0}) \rightarrow H^0(D_{i,0}, \mathcal{O}_{D_{i,0}})$ obtained as the composition of the following maps

$$H^0(C, \Omega_{C|C}^1 \otimes \pi^* T_{\Delta,0}) \rightarrow H^0(C, \Omega_{D_i|D_{i,0}}^1 \otimes \pi^* T_{\Delta,0}) \rightarrow H^0(D_{i,0}, \mathcal{O}_{D_{i,0}}) \simeq \mathbb{C}^l;$$

i.e. $F_1(\Omega) = (\hat{\omega}(p_1), \dots, \hat{\omega}(p_l))$, $F_2(\Omega) = (\hat{\omega}(q_1), \dots, \hat{\omega}(q_l))$. Then we have:

$$\delta\nu(0)(\omega \otimes \eta) = \sum_{i=1}^l \hat{\omega}(p_i) - \hat{\omega}(q_i) \quad (1)$$

(see [Voi88, 5.5 and 5.8] where it is proved in much more generality).

3 The Torelli theorem

Let C be a smooth generic curve of genus $g \geq 4$. Recall that

$$W_{g-1}^i(C) = \{L \in \text{Pic}^{g-1}(C) \mid h^0(L) \geq i+1\},$$

and that its smooth locus is

$$W_{g-1,\text{sm}}^i(C) = \{L \in \text{Pic}^{g-1}(C) \mid h^0(L) = i+1\}.$$

Let's consider a line bundle $L \in W_{g-1,\text{sm}}^1(C)$ and its residue line bundle $\omega_C \otimes L^{-1} \in W_{g-1,\text{sm}}^1(C)$. Note that since C is generic $L^{\otimes 2} \neq \omega_C$.

We fix two basis $\{s_0, s_1\}$, $\{t_0, t_1\}$ of $H^0(C, L)$ and $H^0(C, \omega_C \otimes L^{-1})$ respectively.

Denoting by

$$\mu : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^{-1}) \rightarrow H^0(C, \omega_C)$$

the Petri map, we consider the holomorphic forms $\omega_{ij} = \mu(s_i \otimes s_j)$. The vector space $V = \text{span}\{\omega_{00}, \omega_{01}, \omega_{10}, \omega_{11}\}$ is a four dimensional vector subspace of $H^0(C, \omega_C)$ without base point. We then have a holomorphic map $\phi : C \rightarrow \mathbb{P}V^*$ and $\phi(C)$ lies on the quadric $Q = \{z_{00}z_{11} - z_{01}z_{10} = 0\}$, where z_{ij} are the coordinates of $\mathbb{P}V^* \simeq \mathbb{P}^3$. Furthermore C is birational to $\phi(C)$ by [ACG11, prop. 8.33 p. 834].

Let $\pi : \mathcal{C} \rightarrow B$ be the Kuranishi family of C . Restricting B if necessary, we can suppose that there exist sections $p_i : B \rightarrow \mathcal{C}$, $i = 1, \dots, g-1$, such that $D_t = \sum_{i=1}^{g-1} p_i(t)$ verifies $L \cong \mathcal{O}_C(D_0)$ and $H^0(C_t, \mathcal{O}_{C_t}(D_t)) = 2$ for every $t \in B$, i.e. $L_t = \mathcal{O}_{C_t}(D_t) \in W_{g-1,\text{sm}}^1(C_t)$. We can regard $\{(C_t, L_t)\}_{t \in B}$ as a deformation of (C, L) . Since $2D_t - K_{C_t}$ has degree zero,

we then get a normal function $\nu(t) = AJ_{C_t}(2D_t - K_{C_t})$ (K_{C_t} denotes a canonical divisor of C_t).

Our aim is to prove that the infinitesimal $\delta\nu(0)$ reconstruct the curve $\phi(C) \subset \mathbb{P}^3$.

In order to do that, we have to solve the following problem: given a point $q \in Q$, construct an holomorphic form $\omega_q \in H^0(C, \omega_C)$ and a class of deformation $\xi_q \in H^1(C, T_C)$ uniquely associated to q (up to multiples) such that $\omega_q \otimes \xi_q$ satisfies $\omega_q \cdot \xi_q = 0$.

Remark 3.1. If L_t is a theta characteristic, i.e. $L_t^{\otimes 2} \cong \omega_{C_t}$, then $L_t^{\otimes 2} \otimes \omega_{C_t}^{-1} \cong \mathcal{O}_{C_t}$. So the Abel-Jacobi map is zero.

Remark 3.2. We could have defined the normal function over the relative Brill-Noether variety \mathcal{W}_{g-1}^1 (see [ACG11, chap. XXI]). Its tangent space in fact parametrizes first order deformations of (C, L) that preserves sections of L . We decided to work instead over the Kuranishi family B , to stress the fact that our calculation of $\delta\nu(0)$ depends only on first order deformations of C , which are parametrized by $T_{B,0} \cong H^1(C, T_C)$.

3.1 The form ω_q

Consider the Segre map $S : \mathbb{P}H^0(L) \times \mathbb{P}H^0(\omega_C \otimes L^{-1}) \rightarrow \mathbb{P}V$. The basis chosen in above give projective coordinates x_0, x_1 on $\mathbb{P}H^0(L) \simeq \mathbb{P}^1$ and y_0, y_1 on $\mathbb{P}H^0(\omega_C \otimes L^{-1}) \simeq \mathbb{P}^1$.

If $q \in Q$ then it is the image of a point in $\mathbb{P}^1 \times \mathbb{P}^1$, say $q = S((a_0 : a_1), (b_0 : b_1))$.

Call H the hyperplane of \mathbb{P}^3 containing the two rulings l_1, l_2 passing through q .

With the coordinates z_{ij} on \mathbb{P}^3 we have $l_1 = \{a_1 z_{00} - a_0 z_{10} = a_1 z_{01} - a_0 z_{11} = 0\}$, $l_2 = \{b_1 z_{00} - b_0 z_{01} = b_1 z_{10} - b_0 z_{11} = 0\}$ and $H = \{F = 0\}$, where $F \in H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ is the polynomial $a_1 b_1 z_{00} - a_0 b_1 z_{10} - a_1 b_0 z_{01} + a_0 b_0 z_{11}$. Note that we have

$$S^*(F) = (a_1 x_0 - a_0 x_1)(b_1 y_0 - b_0 y_1).$$

Now we define $s = a_1 s_0 - a_0 s_1$, $t = b_1 t_0 - b_0 t_1$ and

$$\omega_q = \mu(s \otimes t) = a_1 b_1 \omega_{00} - a_0 b_1 \omega_{10} - a_1 b_0 \omega_{01} + a_0 b_0 \omega_{11}.$$

Remark 3.3. 1) If $q = \phi(p)$ we have that $p \in Z(s) \cap Z(t)$, where $Z(s)$ and $Z(t)$ are the zero divisors of s and t .

In fact p is not a base point of $|\omega_C \otimes L^{-1}|$ so $t_0(p)$ and $t_1(p)$ are not both zero. Then $s(p) = 0$ if and only if $s(p)t_0(p) = s(p)t_1(p) = 0$, i.e. $a_1 \omega_{00}(p) - a_0 \omega_{10}(p) = a_1 \omega_{01}(p) - a_0 \omega_{11}(p) = 0$. But this last equation is verified because $q \in l_1$.

In the same way one can check that $t(p) = 0$.

2) An argument similar to the above also yields that $\phi^*(l_{1|\phi(C)}) = Z(s)$.

3.2 The deformation ξ_q

Let's consider the commutative diagram [Grif83, p. 272]

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & \mathbb{P}H^1(\mathcal{O}_C) \\ \downarrow \psi & & \downarrow v \\ \mathbb{P}H^1(T_C) & \longrightarrow & \mathbb{P}Hom^s(H^0(\omega_C), H^1(\mathcal{O}_C)) \cong \mathbb{P}Sym^2(H^1(\mathcal{O}_C)). \end{array} \quad (2)$$

Here Hom^s denote the symmetric homomorphism, φ and ψ are the canonical and bicanonical maps, v is the Veronese map and the bottom map is the natural map, which is an inclusion.

The composition of φ with the projection $\mathbb{P}H^0(\omega_C)^* \rightarrow \mathbb{P}V^*$ is exactly ϕ . We can also consider the composite surjection $Hom^s(H^0(\omega_C), H^1(\mathcal{O}_C)) \rightarrow Hom^s(V, \frac{H^1(\mathcal{O}_C)}{\text{Ann } V})$. So from (2) we obtain the following commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\phi} & \mathbb{P}V^* \\ \downarrow \psi & & \downarrow v \\ \mathbb{P}H^1(T_C) & \longrightarrow & \mathbb{P}Hom^s(V, \frac{H^1(\mathcal{O}_C)}{\text{Ann } V}) \cong Sym^2 V. \end{array} \quad (3)$$

Lemma 3.4. *The variety of rank one transformation in the image of the bottom map in (3) is isomorphic to the quadric Q .*

Proof. The same as in [Grif83, Corollary (5.7)]. \square

We can now proceed to the construction of the deformation ξ_q such that ω_q . We will consider two cases separately.

3.2.1 Case 1. $q \in \phi(C)$

Let $s' \in H^0(C, L)$ and $t' \in H^0(C, \omega_C \otimes L^{-1})$ be such that $\text{span}\{s, s'\} = H^0(C, L)$ and $\text{span}\{t, t'\} = H^0(C, \omega_C \otimes L^{-1})$.

For notational convenience we denote $\mu(s \otimes t) = st$ and similarly for the other cup-products. We define $W = \text{span}\{st, st', s't\} \subset V$ and $\text{Ann } W = \{\xi \in H^1(C, T_C) \mid \xi \cdot W = 0\}$. Note that by Remark 3.3 we have $W \in H^0(C, \omega_C(-p))$ for every $p \in \phi^{-1}(q)$.

Lemma 3.5. *We have $\dim(\text{Ann } W) = 1$.*

Proof. Consider the spaces

$$U = \text{Image}(\mu : W \otimes H^0(C, \omega_C) \rightarrow H^0(C, \omega_C^{\otimes 2})) \text{ and}$$

$$\text{Ann } U = \{\xi \in H^1(C, T_C) \mid H^1(T_C)\langle \xi, U \rangle_{H^0(\omega_C^{\otimes 2})} = 0\}.$$

From the equality

$$H^1(T_C)\langle \xi, \omega\omega' \rangle_{H^0(\omega_C^{\otimes 2})} = H^1(\mathcal{O}_C)\langle \xi \cdot \omega, \omega' \rangle_{H^0(\omega_C)}$$

it follows that $\text{Ann } W = \text{Ann } U$. In order to compute $\dim(\text{Ann } U)$ we set

$$W_1 = \text{span}\{st, s't\}, \quad W_2 = \text{span}\{st, st'\} \text{ and}$$

$$U_i = \text{Image}(\mu : W_i \otimes H^0(C, \omega_C) \rightarrow H^0(C, \omega_C^{\otimes 2})) \text{ for } i = 1, 2.$$

Then $\dim(U_1) = \dim(H^0(C, L) \otimes H^0(C, \omega_C)) - \dim(\ker \mu) = 2g - h^0(\omega_C \otimes L^{-1}) = 2g - 2$, where the second equality follows by the base-point-free-pencil trick [ACGH85, p. 126].

In the same way we find $\dim(U_2) = 2g - 2$.

Now we calculate $\dim(U_1 \cap U_2)$. If $\sigma \in U_1 \cap U_2$ then $(st)\omega_1 + (st')\omega_2 = \sigma = (st)\eta_1 + (s't)\eta_2$, with $\eta_i, \omega_i \in H^0(C, \omega_C)$ for $i = 1, 2$. By Remark (3.3) we know that $p \in Z(s) \cap Z(t)$, where $q = \phi(p)$. Then $Z(s) = D_1 + p$ and $Z(t) = D_2 + p$ and $\sigma \in H^0(C, \omega_C^{\otimes 2}(-D_1 - D_2 - p)) = H^0(C, \omega_C(p))$.

So we have that $\dim(U_1 \cap U_2) = \dim H^0(C, \omega_C(p)) = g$ and $\dim U = 2g - 2 + 2g - 2 - g = 3g - 4$. By duality we have $\dim(\text{Ann } U) = 1$. \square

Let $\theta_p \in H^1(C, T_C)$ be the Schiffer variation in $p \in C$. Since $\ker \theta_p = H^0(C, \omega_C(-p))$ (see [Grif83, p. 275]) we have $\theta_p \in \text{Ann } W$. If $p' \neq p$ is another point such that $\phi(p') = q$ then also $\theta_{p'} \in \text{Ann } W$ and by the Lemma there exist $\lambda \in \mathbb{C}$ such that $\theta_p = \lambda \theta_{p'}$ as elements of $\text{Hom}^s\left(V, \frac{H^1(\mathcal{O}_C)}{\text{Ann } V}\right)$. So it is natural to define $\xi_q = \theta_p$.

Remark 3.6. Although the space W depends on the choice of the sections s' and t' , the space $\text{Ann } W$ does not. So indeed we have that the deformation ξ_q doesn't depend on the choice of s' and t' .

3.2.2 Case2. $q \in Q \setminus \phi(C)$

If we denote $F = L \oplus (\omega_C \otimes L^{-1})$ we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_C & \xrightarrow{(t, -s)} & F^* & \xrightarrow{\begin{pmatrix} s \\ t \end{pmatrix}} & \mathcal{O}_C \longrightarrow 0 \\ & & \downarrow \omega_q & & \downarrow \omega_q & & \downarrow \omega_q \\ 0 & \longrightarrow & \mathcal{O}_C & \xrightarrow{(s, t)} & F & \xrightarrow{\begin{pmatrix} t \\ -s \end{pmatrix}} & \omega_C \longrightarrow 0. \end{array} \quad (4)$$

The first row defines an element of $\text{Ext}^1(\mathcal{O}_C, T_C)$. We call $\delta : H^0(C, \mathcal{O}_C) \rightarrow H^1(C, T_C)$ and $\delta' : H^0(C, \omega_C) \rightarrow H^1(C, \mathcal{O}_C)$ the coboundaries of the first and second row. Let's

define $\xi_q = \delta(1)$ and recall the map δ' is given by cupping with ξ_q . The form ω_q has a lifting to $H^0(C, F) \simeq H^0(C, L) \oplus H^0(C, \omega_C \otimes L^{-1})$ (for example $(s, 0)$), so by exactness $0 = \delta'(\omega_q) = \omega_q \cdot \xi_q = 0$.

3.3 Computation of $\delta\nu$

Let's call for brevity $\mathbb{P} = \mathbb{P}H^1(C, T_C)$, and consider

$$\Sigma = \{((\xi), (\omega)) \in \mathbb{P} \times \mathbb{P}H^0(C, \omega_C) \mid \xi \cdot \omega = 0\},$$

where we call (ξ) the class of ξ in \mathbb{P} and similarly for (ω) . Inside Σ we have the variety $X = \{((\xi_q), (\omega_q)) \in \mathbb{P} \times \mathbb{P}V \mid q \in Q\}$. Under the isomorphism $\mathbb{P}V \simeq \mathbb{P}V^*$, the form ω_q correspond to the projective tangent space $T_{Q,q}$ (see (3.1)). Then

$$X \simeq \{(q, T_{Q,q}) \in Q \times \mathbb{P}V^*\},$$

and the latter is isomorphic to Q . Let's call π_1 and π_2 the projections from $\mathbb{P} \times \mathbb{P}V^*$ onto \mathbb{P} and $\mathbb{P}V^*$ respectively. Then $\delta\nu(0)$ can be thought as a map

$$\delta\nu(0) : (\pi_1^* \mathcal{O}_{\mathbb{P}}(-1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}V^*}(-1))|_Q \rightarrow \mathcal{O}_Q,$$

and taking the dual map we get an element of $H^0(Q, \pi_1^* \mathcal{O}_{\mathbb{P}}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}V^*}(1))|_Q$. By abuse of notation we continue to call this section $\delta\nu(0)$.

In the remaining part of the paper we will prove the following

Proposition 3.7. *We have*

$$\delta\nu(0)(q) = 0 \text{ if } q \in \phi(C) \tag{5}$$

$$\delta\nu(0)(q) \neq 0 \text{ if } q \in Q \setminus \phi(C). \tag{6}$$

In other words, the section $\delta\nu(0) \in H^0(Q, (\pi_1^ \mathcal{O}_{\mathbb{P}}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}V^*}(1))|_Q)$ vanishes on $\phi(C)$.*

3.3.1 Case 1. $q \in \phi(C)$

Recall that for (U, z) a coordinate chart centered in p , locally in U we have that $\theta_p = \frac{\bar{\partial}\rho(z)}{z} \frac{\partial}{\partial z}$, where ρ is a bump function in p . In this coordinate chart we can also write $\omega_q = z f(z) dz$ for some holomorphic function f . So $\xi_q \cdot \omega_q = \bar{\partial}(f\rho)$.

Set $Z(s) = \sum_{i=1}^{g-1} p_i$ and $Z(t) = \sum_{i=1}^{g-1} q_i$. Since $p \in Z(s) \cap Z(t)$ we can suppose that $p = p_1 = q_1$. Then by Theorem 2.1 we have

$$\delta\nu(0)(\omega_q \otimes \xi_q) = \sum_{i=1}^{g-1} f(p_i) \rho(p_i) - f(q_i) \rho(q_i) = 0,$$

where the second equality follows by the fact that ρ is a bump function.

3.3.2 Case 2. $q \in Q \setminus \phi(C)$

Let $\Delta \subset B$ be the unit disc in \mathbb{C} such that $T_{\Delta,0} = \text{span}\{\xi_q\}$ (here we are using that $T_{B,0} \simeq H^1(C, T_C)$). By pull-back we can suppose that our family is $\pi : \mathcal{C} \rightarrow \Delta$. Recall from Section 2 that we have the exact sequence

$$H^0(C, \Omega_{\mathcal{C}|C}^1 \otimes \pi^* T_{\Delta,0}) \simeq H^0(C, \Omega_{\mathcal{C}|C}^1) \otimes T_{\Delta,0} \xrightarrow{\gamma} H^0(C, \omega_C) \otimes T_{\Delta,0} \xrightarrow{\delta} H^1(C, \mathcal{O}_C).$$

Remember that we have constructed $s \in H^0(L)$ and $t \in H^0(K_C \otimes L^{-1})$ and that $\omega_q = \mu(s \otimes t)$, $Z(s) = \sum_{i=1}^{g-1} p_i$, $Z(t) = \sum_{i=1}^{g-1} q_i$.

We write, locally near p_i , $\omega_q = f_i(z_i)dz_i$, and near q_i , $\omega_q = \tilde{f}_i(w_i)dw_i$. Now since $\delta(\omega_q \otimes \xi_q) = \xi_q \cdot \omega_q = 0$, $\omega_q \otimes \xi_q$ lifts to $\Omega = \hat{\omega}_q \otimes \xi_q$, with $\hat{\omega}_q \in H^0(C, \Omega_{\mathcal{C}|C}^1)$.

Then we have $\hat{\omega}_q = f_i(z_i)dz_i + g_i(z_i)dz_i$ near p_i and $\hat{\omega}_q = \tilde{f}_i(w_i)dw_i + \tilde{g}_i(w_i)dw_i$ near q_i .

With the notation of section 2 we have $D_{1,0} = \sum_{i=1}^{g-1} p_i$ and $D_{2,0} = \sum_{i=1}^{g-1} q_i$ and

$$F_1(\hat{\omega}_q \otimes \xi_q) = (g_1(p_1), \dots, g_{g-1}(p_{g-1})) \text{ and}$$

$$F_2(\hat{\omega}_q \otimes \xi_q) = (\tilde{g}_1(q_1), \dots, \tilde{g}_{g-1}(q_{g-1})),$$

because $f_i(p_i) = \tilde{f}_i(q_i) = 0$ for $i = 1, \dots, g-1$. So by (1) we have

$$\delta\nu(0)(\omega_q \otimes \xi_q) = \sum_{i=1}^{g-1} g_i(p_i) - \tilde{g}_i(q_i). \quad (7)$$

Lemma 3.8. *There exist $c \in \mathbb{C}$ such that $g_i(p_i) = c$ and $\tilde{g}_i(q_i) = c+1$ for $i = 1, \dots, g-1$.*

Proof. We have the commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T_C & \longrightarrow & T_{\mathcal{C}|C} & \longrightarrow & \mathcal{O}_C \longrightarrow 0 \\ & & \downarrow \omega_q & & \downarrow \omega_q & & \downarrow \omega_q \\ 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \Omega_{\mathcal{C}|C}^1 & \longrightarrow & \omega_C \longrightarrow 0 \\ & & \downarrow r & & \downarrow r & & \downarrow r \\ 0 & \longrightarrow & \mathcal{O}_{Z(\omega_q)} & \xrightarrow{i} & \Omega_{\mathcal{C}|Z(\omega_q)}^1 & \longrightarrow & \omega_{C|Z(\omega_q)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}, \quad (8)$$

where $\mathcal{O}_{Z(\omega_q)} \simeq \bigoplus_{i=1}^{g-1} \mathbb{C}_{p_i} \oplus \bigoplus_{i=1}^{g-1} \mathbb{C}_{q_i}$. It gives the following diagram in cohomology

$$\begin{array}{ccccc} \mathbb{C} & \longrightarrow & H^0(C, \Omega_{C|C}^1) & \longrightarrow & H^0(C, \omega_C) \\ \downarrow r & & \downarrow r & & \\ \mathbb{C}^{2g-2} & \xrightarrow{i} & H^0(Z(\omega_q), \Omega_{C|Z(\omega_q)}^1) & & \end{array} .$$

It is clear that $r(\hat{\omega}_q) = i((g_1(p_1), \dots, g_{g-1}(p_{g-1}), \tilde{g}_1(q_1), \dots, \tilde{g}_{g-1}(q_{g-1})))$. Note that the association $\omega_q \mapsto (g_1(p_1), \dots, g_{g-1}(p_{g-1}), \tilde{g}_1(q_1), \dots, \tilde{g}_{g-1}(q_{g-1})) \in \mathbb{C}^{2g-2}$ depends on the choice of a lifting $\hat{\omega}_q$, so it is well defined modulo $r(\mathbb{C})$.

Under the isomorphism $F^* \simeq T_{C|C}$ diagram (8) becomes

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T_C & \longrightarrow & F^* & \longrightarrow & \mathcal{O}_C \longrightarrow 0 \\ & & \downarrow \omega_q & & \downarrow \omega_q & & \downarrow \omega_q \\ 0 & \longrightarrow & \mathcal{O}_C & \xrightarrow{(s,t)} & F & \xrightarrow{\begin{pmatrix} t \\ -s \end{pmatrix}} & \omega_C \longrightarrow 0 \\ & & \downarrow r & & \downarrow r & & \downarrow r \\ 0 & \longrightarrow & \mathcal{O}_{Z(\omega_q)} & \xrightarrow{i} & F|_{Z(\omega_q)} & \longrightarrow & \omega_{C|Z(\omega_q)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array} \tag{9}$$

In cohomology $\omega_q = st$ lifts to $(s, 0) \in H^0(C, F)$. It restricts to $(s|_{Z(\omega_q)}, 0) \in H^0(Z(\omega_q), F|_{Z(\omega_q)})$, which lifts to $(0, \dots, 0, 1, \dots, 1) \in \mathbb{C}^{g-1} \oplus \mathbb{C}^{g-1}$.

So in this case we have that $\omega_q \mapsto (0, \dots, 0, 1, \dots, 1) \in \mathbb{C}^{2g-2}$.

Comparing with what we found above we conclude

$$(g_1(p_1), \dots, g_{g-1}(p_{g-1}), \tilde{g}_1(q_1), \dots, \tilde{g}_{g-1}(q_{g-1})) = (0, \dots, 0, 1, \dots, 1) \bmod r(\mathbb{C}),$$

which is the thesis. \square

Corollary 3.9. *For $q \in Q \setminus \phi(Q)$ we have $\delta\nu(0)(\omega_q \otimes \xi_q) \neq 0$.*

Proof. Use Lemma (3.8) and (7). \square

This completes the proof of Proposition 3.7. We can deduce the following

Corollary 3.10. *The infinitesimal invariant reconstruct the couple (C, L) .*

Proof. It is clear that $\delta\nu(0)$ reconstructs C , since ϕ is birational onto $\phi(C)$. To recover L , it suffices to recall that by remark 3.3 we have $\phi^*(l_{1|\phi(C)}) = Z(s)$ (notations as in section 3.1), and that obviously $\mathcal{O}_C(Z(s)) \cong L$. \square

Remark 3.11. When $g = 4$ we have that $\phi(C)$ is the canonical image of C . We thus obtain Griffiths' result [Grif83, pp. 298-302].

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